Relations and Functions

Computer Mathematics I

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ORDERED PAIRS

An **ordered pair** is written as (a,b) where a and b are called the components of the ordered pair(a,b).

- The ordered pair(a,b) is different from the set a,b.
- Unlike sets, the ordered pair(a,b) is not the same as (b,a).
- Two ordered pair(a,b) and (c,d) are equal only when a=c and b=d.

RELATIONS

The **Cartesian product** of two sets A,B denoted by A×B, is the set of all ordered pairs(a,b) with a∈A, b∈B.
Example:
{1,2} × {3} = {(1,3),(2,3)}

A **binary relation** on two sets A,B is a subset of A×B.
Example:
{(1,b),(1,c),(3,d),(9,d)} is a binary relation on {1,3,9} and {b,c,d}.
{(n,m): n,m ∈ ℕ and n<m } is a binary relation on ℕ × ℕ
An **ordered n-tuple** is written as $(a_1, \ldots, a_n)$ where a_i is the Ith component of $(a_1, \ldots, a_n)$.
The **n-fold Cartesian product** of the set $A_1, \ldots, A_n$, denoted by $A_1 \times \ldots \times A_n$ is the set of all ordered n-tuples with $a_i \in A_i$ for i=1,...,n.
An **n-ary** relation on sets $A_1, \ldots, A_n$ is a subset of $A_1 \times \ldots \times A_n$.

FUNCTIONS

A **function** from a set A to a set B,denoted f: $A \to B$ is a binary relation R on A and B where for each element a∈A, there is exactly one ordered pair in R with first component a.
Example: let A={1,2} and B={3,4}
R_1= {(1,3),(2,3)}
R_2= {(1,3),(2,4)}
are functions.
But R_3= {(1,3)} is not a function.

FUNCTIONS

A function f: $A \to B$, we call A the **domain** of f and B is the **codomain** of f.
If a is any element of A we write f(a) for that element b of B such that (a,b)$\in$f.
f(a) is called the **image** of a under f.

The range of f is the image of its domain, denoted by

$$f(A)=\{b|\text{there is a in A such that } b=f(a)\}$$

For $A' \subseteq A$, f(A') = {f(a): a$\in A'$ } is called the image of A' under f.

Example of functions:
The integer addition(+) is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

FUNCTIONS

A function f: $A \rightarrow B$ is **one-to-one** if for any two distinct elements $a, a' \in$ A, $f(a) \neq f(a')$

A function f: $A \rightarrow B$ is **onto** B if each element of B is the image under f of some element of A.

A function f $A \rightarrow B$ is a **bijection** between A and B if it is both one-to-one and onto B.

Example: let A={1,2} and B={3,4}
R_1= {(1,3),(2,4)} is one-to-one and onto B, hence R_1 is bijective
R_2= {(1,3),(2,3)} is neither one-to-one nor onto B.

INVERSE OF A FUNCTION

The **inverse** of a binary relation $R \subseteq A \times B$, denoted $R^{-1} \subseteq B \times A$ is the relation {(b,a): (a,b)∈R}.
For example, let R_2= {(1,3),(2,3)} then R_2^{-1}= {(3,1),(3,2)}.

The inverse of a function need not be a function.
For example, R_2= {(1,3),(2,3)} is a function but R_2^{-1}= {(3,1),(3,2)} is not a function.
A function f: $A \rightarrow B$ may fail to have an inverse if there is some element $b \in B$ such that $f(a) \neq b$ for all $a \in A$.

$f^{-1}(f(a)) = a$ for each $a \in A$ and $f(f^{-1}(b)) = b$ for each $b \in B$.

COMPOSITIONS

For binary relations $Q \subseteq A \times B$ and $R \subseteq B \times C$, the **composition** $Q \circ R$ is defined by

$$Q \circ R = \{(a,c) : \exists b \in B \text{ such that } (a,b) \in Q \text{ and } (b,c) \in R\}$$

The composition of two functions f: $A \to B$ and g: $B \to C$ is a function h from A to C such that h(a)=g(f(a)) for each $a \in A$.

Example:
Let $R_1 = \{(1,a),(2,a)\}$ and $R_2 = \{(a,3),(b,4)\}$
$R_1 \circ R_2 = \{(1,3),(2,3)\}$

EXERCISES

Let R={(a,b),(a,c),(c,d),(a,a),(b,a)}
What is $R \circ R$?
What is R^{-1} ?
Is R, $R \circ R$, or R^{-1} a function ?