Special Types of Binary Relations

Computer Mathematics I

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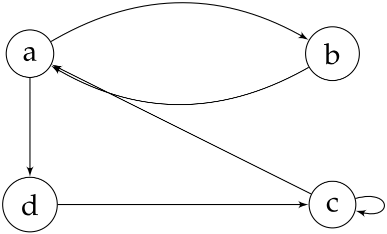
DIRECTED GRAPH

An relation R $\subseteq A \times A$, for any set A, can be represented as a **directed graph**.

- A node of the directed graph is represented by a small circle to represent each element of A.
- An arrow is drawn from a to b if and only if (a,b)$\in$R.
- The arrows are edges of the directed graph.
- From a node of a graph to another there is either no edge, or one edge.

For example, R ={(a,b), (b,a), (a,d), (d,c), (c,c), (c,a)}



TYPES OF RELATIONS

A relation R $\subseteq A \times A$ is **reflexive** if (a,a)∈R for each a∈A.

Example:
Let A={1,2}
R= {(1,1), (1,2), (2,2)} is reflexive.
R = {(1,2), (2,2)} is not reflexive.

TYPES OF RELATIONS

A relation $R \subseteq A \times A$ is **symmetric** if (b,a)∈R whenever (a,b)∈R. Symmetric relation can be represented by **undirected graph**.

Example:
Let A={1,2}
R = {(1,2), (2,1), (2,2)} is symmetric.
R= {(1,1), (1,2), (2,2)} is not symmetric.

TYPES OF RELATIONS

A relation R $\subseteq A \times A$ is **antisymmetric** if whenever(a,b)$\in$R and a and b are distinct, then (b,a)$\notin$R.

Example:
Let A={1,2}
R= {(1,1), (1,2), (2,2)} is antisymmetric.
R = {(1,2), (2,1), (2,2)} is not antisymmetric.

TYPES OF RELATIONS

A relation R $\subseteq A \times A$ is **transitive** if whenever (a,b)$\in$R and (b,c)$\in$R then (a,c)$\in$R.

Example:
Let A={1,2,3}
R= {(1,1), (1,2), (2,2)} is not transitive. R = {(1,2), (1,3), (2,3)} is transitive.

EQUIVALENCE

A relation $R \subseteq A \times A$ is an **equivalence relation** if R is reflexive, symmetric and transitive.

Let $\equiv$ be an equivalent relation on a set A. Then for each a in A, **the equivalence class of a with respect to** $\equiv$ is denoted by $[a]_{\equiv}$ and is defined formally by

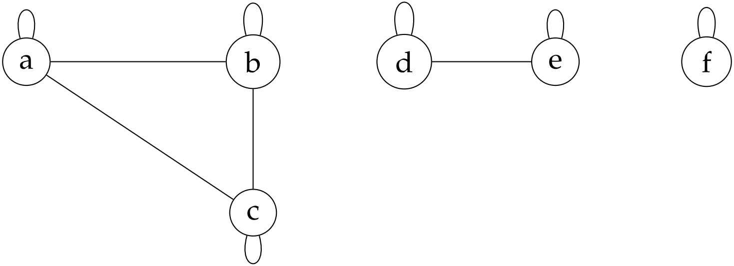
$$[a]_{\equiv} = \{b \mid (a,b) \in R\}$$

When the context $\equiv$ is clear, we simply write [a] for $[a]_{\equiv}$.

A representation of an equivalent relation $R \subseteq A \times A$ as an undirected graph consists of a number of **"clusters"** where within clusters each pair is connected by a line. The set of nodes in a cluster is an equivalence class.

EQUIVALENCE

Example:
Let A={a,b,c,d,e,f}
R $\subseteq A \times A$ and R is represented as the following undirected graph.



$[a]_R = [b]_R = [c]_R = \{a,b,c\}$ $[d]_R = [e]_R = \{d,e\}$ $[f]_R = \{f\}$

EQUIVALENCE AND MODULO

Property Let $\equiv$ be an equivalent relation on a set A. Then for any two elements a,b in A, either [a]=[b] or [a], [b] are disjoint.

Let R be an equivalent relation on a set A. Define **A modulo R** to be the set

$$A/R = \{[a]_R \mid a \in A\}$$

Theorem 1 Let R $\subseteq A \times A$ be an equivalent relation. Then A modulo R is a partition of A.

Example:(from previous example)

$A/R = \{[a]_R, [d]_R, [f]_R\} = \{[b]_R, [e]_R, [f]_R\}$

$= \{\{a,b,c\},\{d,e\},\{f\}\}$

PARTIAL AND TOTAL ORDERS

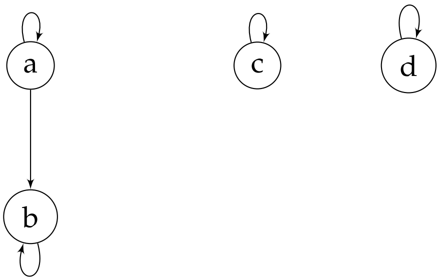
A relation that is reflexive, transitive and antisymmetric is called a **partial order**.

A partial order R $\subseteq A \times A$ is called a **total order** if for all a,b $\in$ A, either (a,b) $\in$ R or (b,a) $\in$ R.

A **Path from** a **to** b in a binary relation R $\subseteq A \times A$ is a sequence $(a_1, \ldots, a_n)$, n $\geq$ 1 such that a = a_1, b = a_n, for each i = 1,...n-1, (a_i, a_{i+1}) $\in$R.

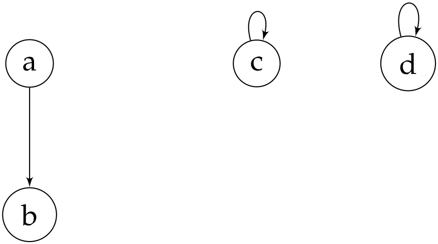
A path $(a_1, \ldots, a_n)$ is a **cycle** if $a_1 = a_n$ and all a_i's are distinct.

Example of Partial and Total orders



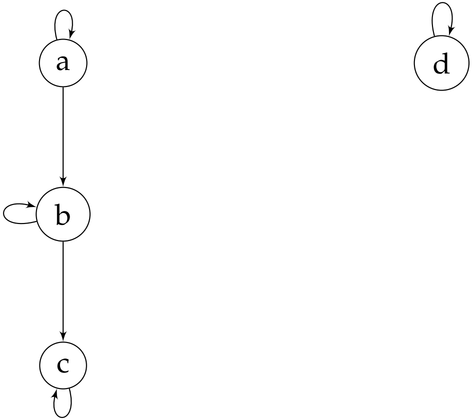
R_0 is reflexive, transitive, and antisymmetric. Hence it's a partial order. It's NOT a total order.

Example of Partial and Total orders



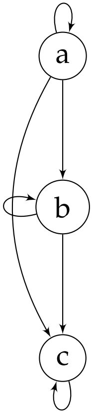
R_1 is not reflexive. Hence it's NOT a partial order.

Example of Partial and Total orders



R_2 is not transitive. Hence it's NOT a partial order.

Example of Partial and Total orders



R_3 is reflexive, symmetric, and transitive. Hence it's a partial order.

For all a,b $\in$ A, either (a,b) $\in$ R or (b,a) $\in$ R then it's a total order.

EXERCISES

Let A be a nonempty set and let R $\subseteq A \times A$ be the empty set. Which properties does R have?

1. Reflexive.
2. Symmetry.
3. Antisymmetry.
4. Transitivity.

EXERCISES

Let R $\subseteq A \times A$ be a binary relation as defined below. In which cases is R a partial order? a total order?

1. A = the positive integers; (a,b)$\in$R if and only if b is divisible by a.
2. A = $\mathbb{N}$; (a,b)$\in$R if and only if b=a or b= a+1.