Ch2: Loop Invariants

305233, 305234

Algorithm Analysis and Design

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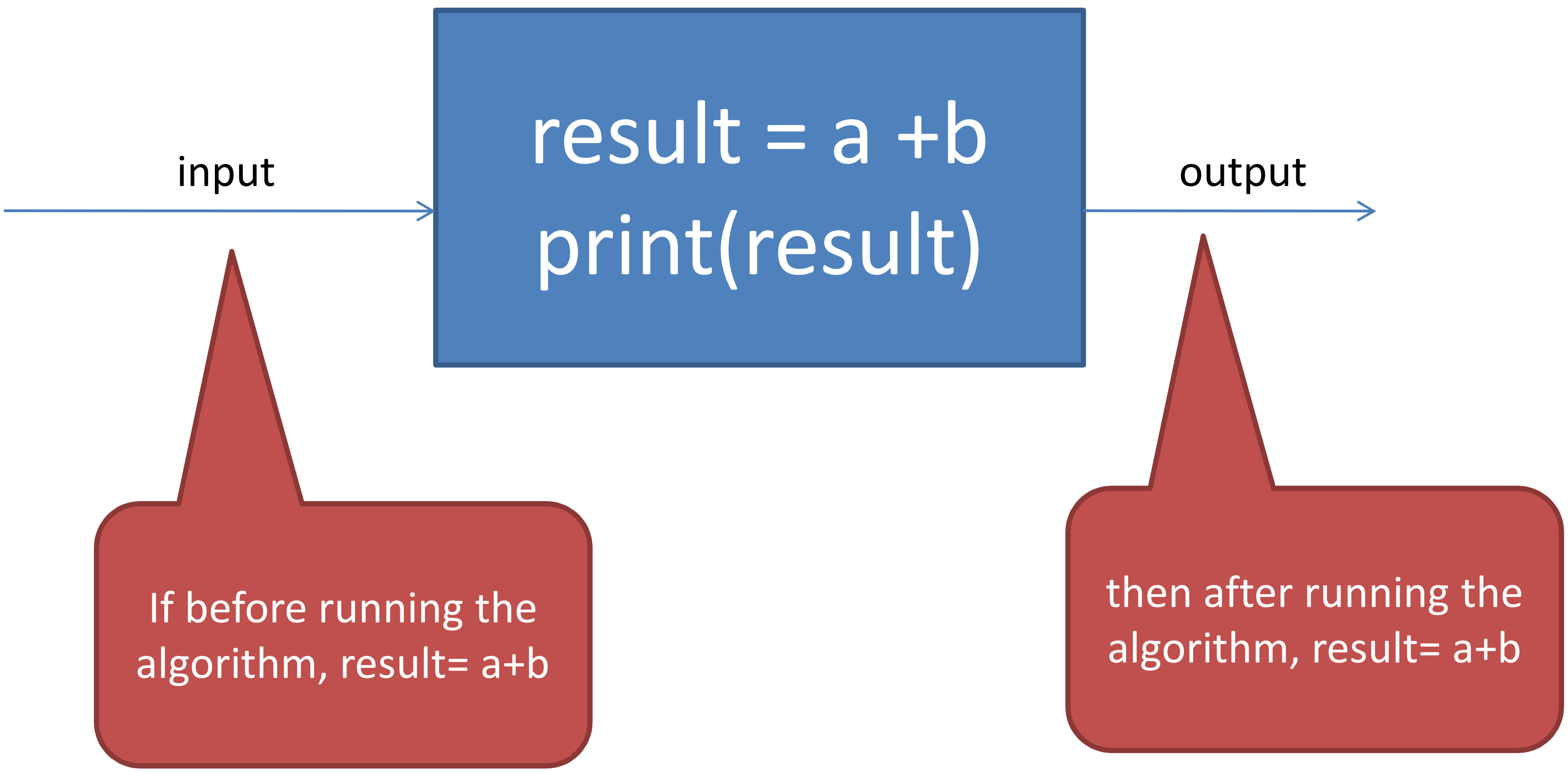
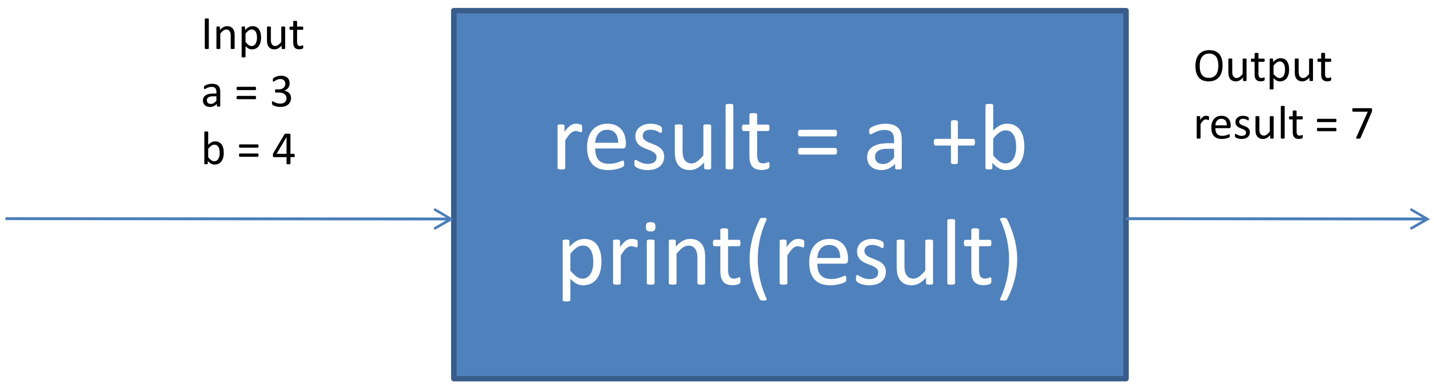
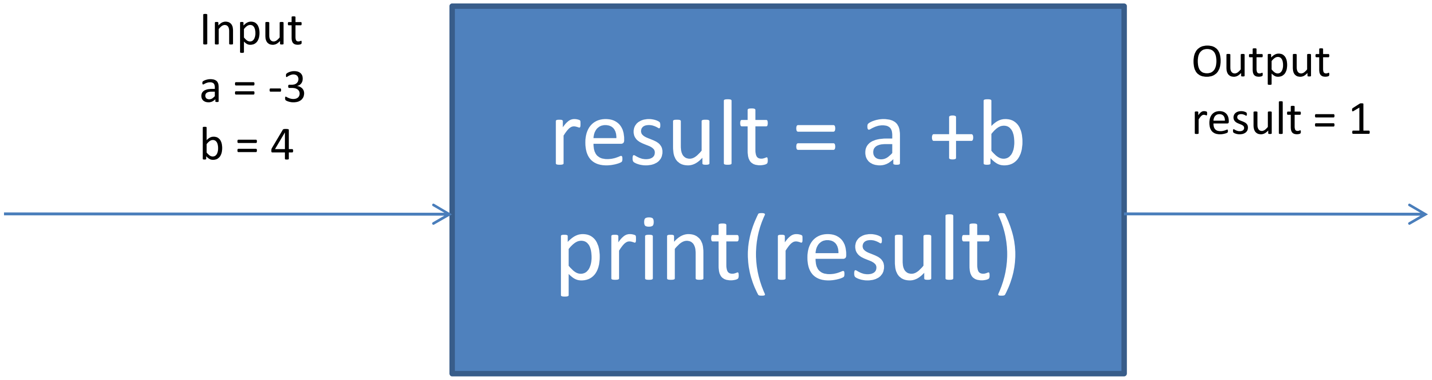
Accuracy of an Algorithm



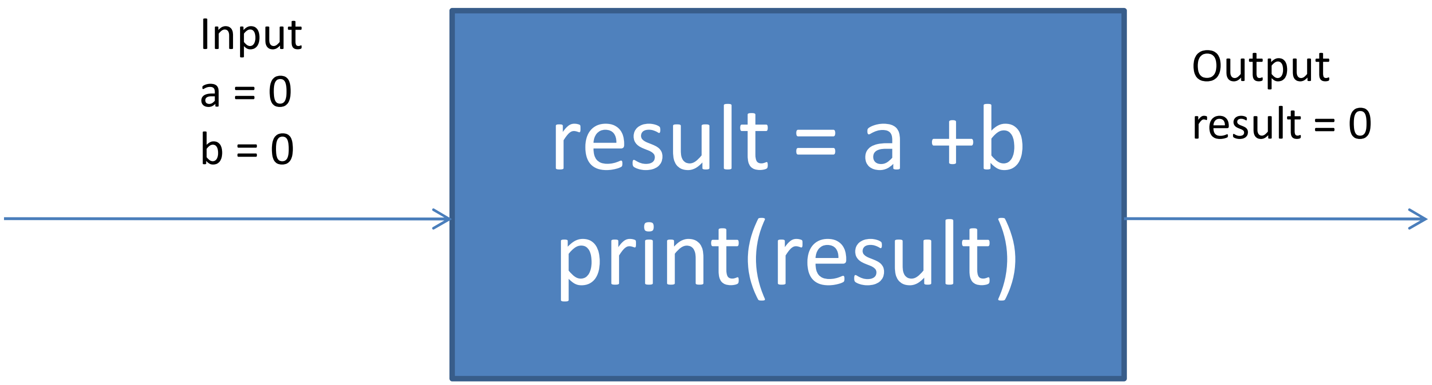
Accuracy of an Algorithm



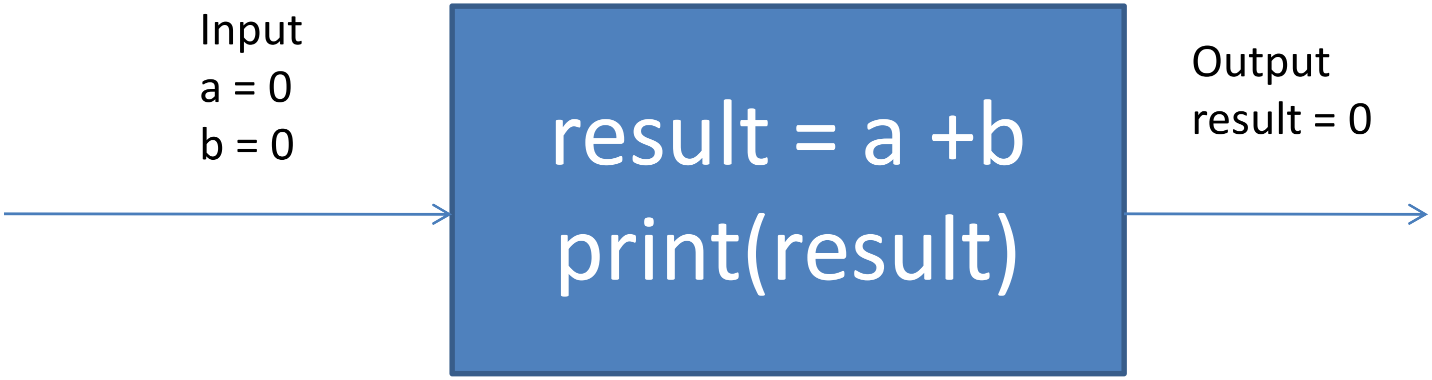
Accuracy of an Algorithm



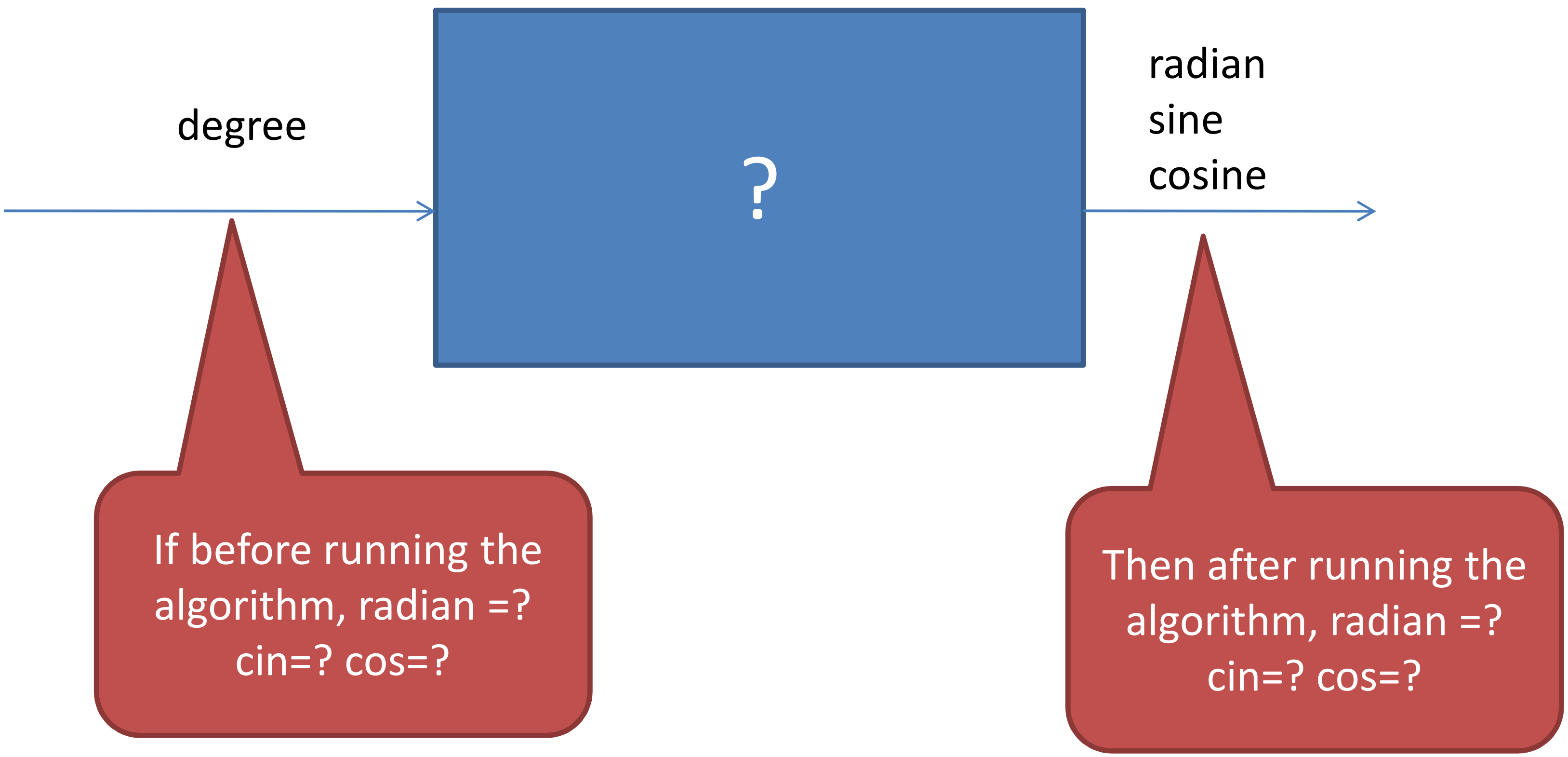
Accuracy of an Algorithm



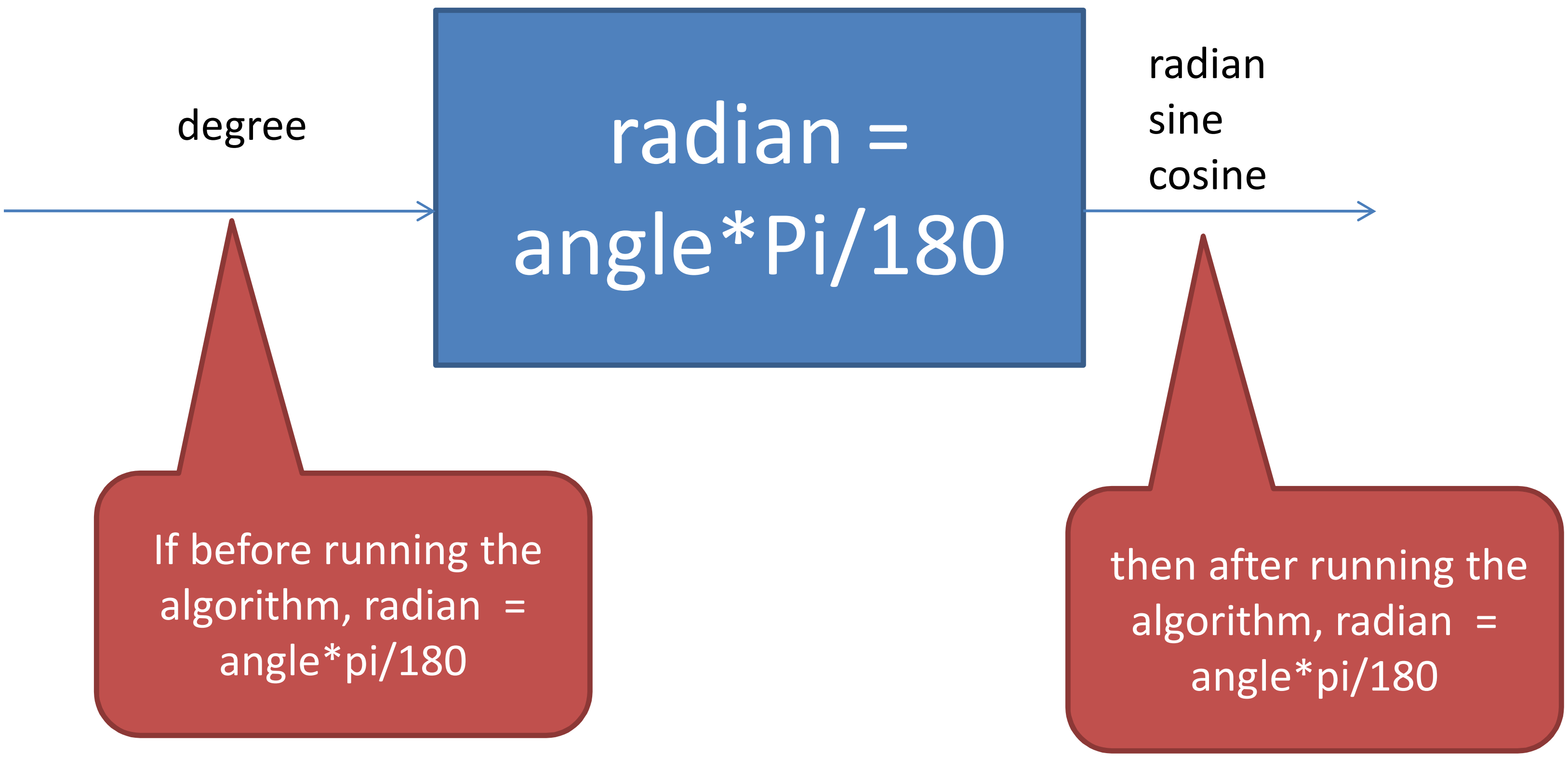
Accuracy of an Algorithm



Example: Accuracy of an Algorithm



Example: Accuracy of an Algorithm



Example

```
import math
a = float(input("Enter an angle in degrees: "))
r = a*(22/7)/180
print("%f degrees = %.2f radians and sin(%f) = %.2f and cos(%f) = %.2f" %
      (a,math.pi(a),math.sin(r),math.cos(r)))
```

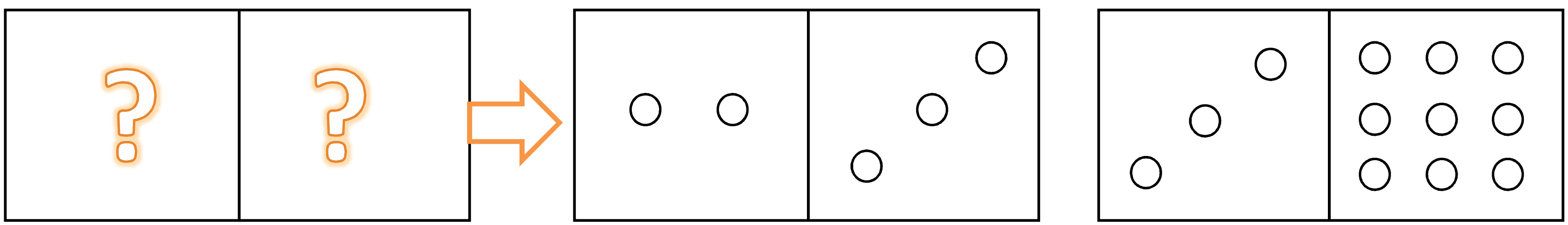
```
degree = int(input("Enter an angle in degrees: "))
import math
radian = (degree*math.pi)/180
sin = math.sin(radian)
cosine = math.cos(radian)
print("%d degrees = %.2f radians and sin(%d) = %.2f and cos(%d) = %.2f"%
      (degree,radian,degree,sin,degree,cosine))
```

Example

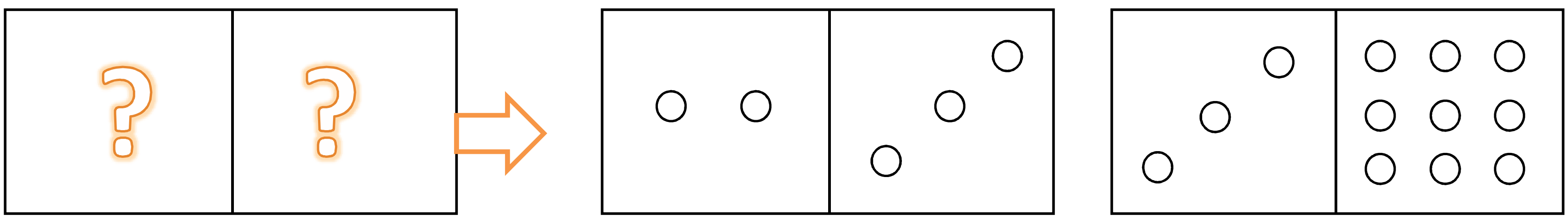
```
import math
pi = 3.14
angle = int(input("Enter an angle in degrees: "))
radian = angle*(pi)/180
print("%d degrees = %.2f radian" % angle, radian)
```

```
number = math.sin(input('Enter an angle in degrees:'))
Ra = (number*(math.pi))/180
math.sin = number
```

Example: Accuracy of an Algorithm



Example: Accuracy of an Algorithm



Domino tile
(left, right)

left/right =2
or
left/right = 9

Place Left / Right/ None

If before running the algorithm, right of input = left of given or left of input = right of given

then after running the algorithm, right of input = left of given or left of input = right of given

Example

```
print("Enter your domino tile: ")
x = int(input())
y = int(input())

if x==2 and y==9 or x==9 and y==2:
        print("Place your block the righ.")
        print("Place your block the left.")
elif x==9 or y==9:
       print("Place your block the righ.")
elif x==2 or y==2:
       print("Place your block the left.")
else:
       print("Place your block the tile.")
```

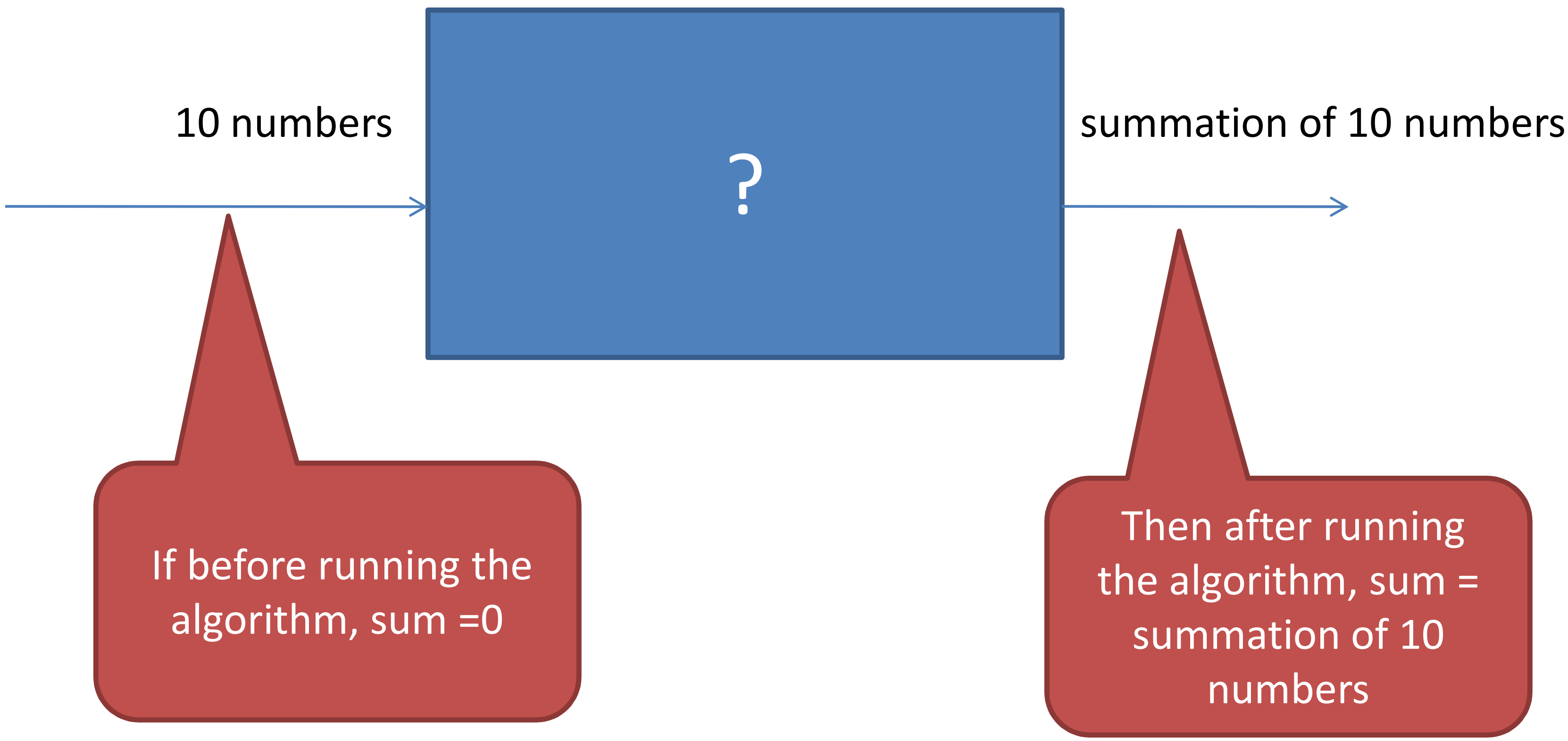
Example

```
x = int(input("Enter your domino title: = " ))
y = int(input("Enter your domino title: = " ))
if x==2 and y==9:
    print("place your block on the left amd right")
elif x==2 and y != 9:
         print("place your block on the left ")
elif x!=2 and y == 2:
         print("place your block on the left ")
elif x==2 and y ==2:
         print("place your block on the right ")
elif x==9 and y == 9:
         print("place your block on the left ")
elif x==9 and y == 2:
         print("place your block on the left amd right")
elif x!=2 and y == 9:
         print("place your block on the right ")
elif x!=9 and y == 9:
         print("place your block on the right ")
elif x!=9 and y == 2:
         print("place your block on the left ")
elif x!=9 and y != 2:
         print("cannot place your block tite  ")
elif x!=9 and y != 9:
         print("cannot place your block tite ")
elif x!=2 and y != 2:
         print("cannot place your block tite  ")
elif x!=2 and y != 9:
         print("cannot place your block tite ")
```

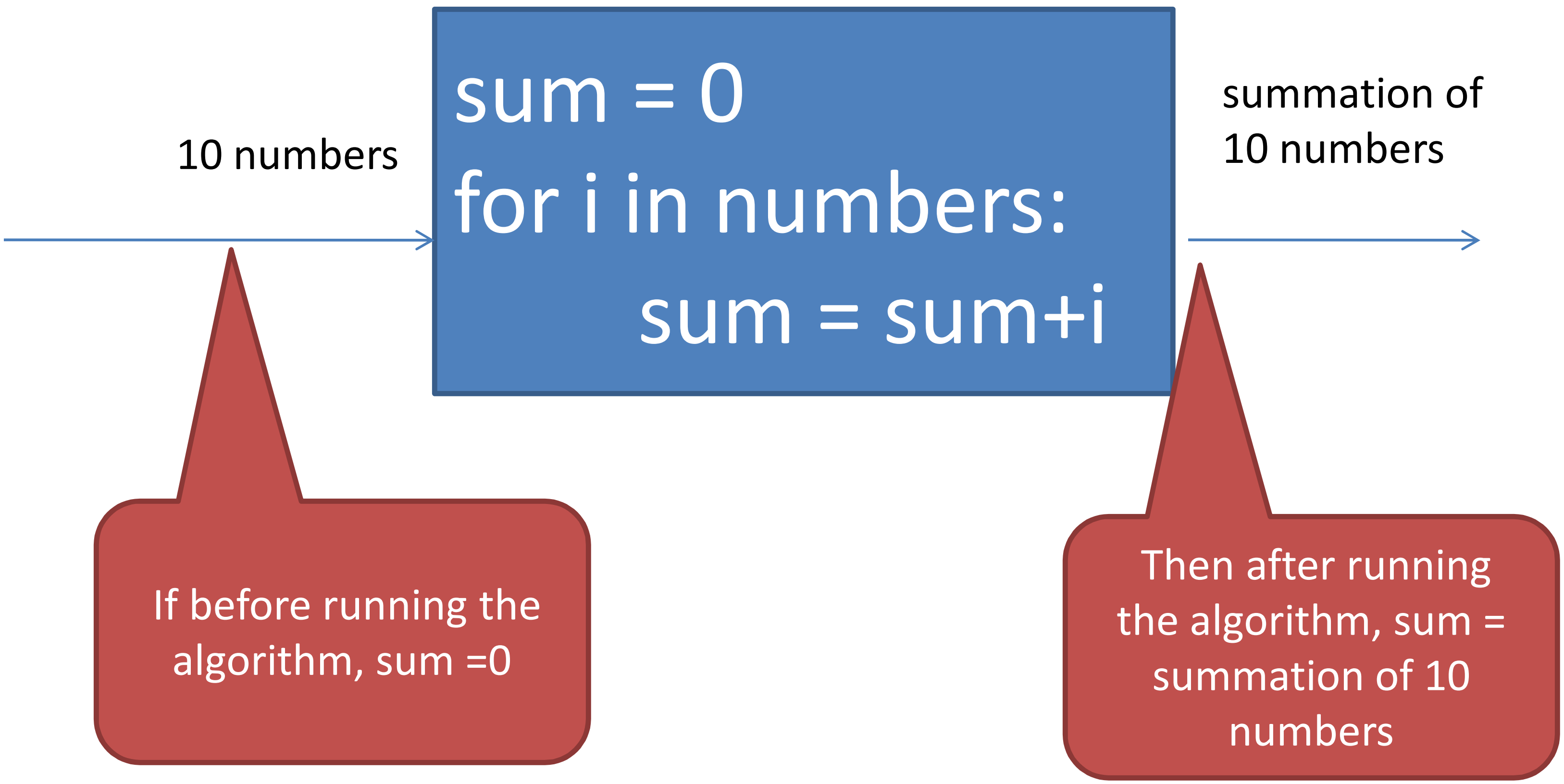
Example

```
print(int(input("Enter your domino tile : ")))
left = (int(input(" ")))
right = (int(input(" ")))

if left == 2 or right == 9:
    print("Place your block on the left.")
    print("Place your block on the right.")

elif right == 9:
    print("Place your block on the right.")

elif left == 2:
    print("Place your block on the left.")

else:
    print("Cannot place your domino tile.")
```

Example: Accuracy of an Algorithm



Example: Accuracy of an Algorithm



Example: Accuracy of an Algorithm

Round 1

[1,2,3,4,5,6,7,8,9,10]

```
sum = 0
for i in numbers:
        sum = sum+i
```

summation of
0 +1

If before running loop 1, sum =0

Then after running loop 1, sum = 1

Example: Accuracy of an Algorithm

Round 2

[1,2,3,4,5,6,7,8,9,10]

```
sum = 0
for i in numbers:
        sum = sum+i
```

summation of
0 +1 + 2

If before running loop 2, sum =1

Then after running loop 2, sum = 3

Example: Accuracy of an Algorithm

Round 3

[1,2,3,4,5,6,7,8,9,10]

```
sum = 0
for i in numbers:
        sum = sum+i
```

summation of
0 +1 + 2+3

If before running loop 3, sum =3

Then after running loop 3, sum = 6

Example: Accuracy of an Algorithm

Round 10

[1,2,3,4,5,6,7,8,9,10]

```
sum = 0
for i in numbers:
        sum = sum+i
```

summation of
0 +1 + 2+3+4+5
+6+7+8+9+10

If before running loop 10, sum =45

Then after running loop 10, sum = 55

Before running loop N, we have sum value of loop N-1

What is a Loop Invariant?

- An **loop invariant** is a formal statement of a properties of variables in an algorithm which holds true just before and after each iteration of running the **loop**.
- Similar to mathematical induction where the initialization is proving a base case and the maintenance is proving an inductive step.

proofs of a Loop Invariant

- **Initialization**
  - It is true prior to the first iteration of the loop.
- **Maintenance**
  - If it Is true before an iteration of the loop, it remains true before the next iteration.
- **Termination**
  - When the loop terminates, the invariant gives a useful property that helps shows that the algorithm is correct.

Example loop invariants with summation

```
sum = 0
for  i=1 to length[A]
        sum = sum + A[i]
```

What is a loop invariant for this code?

A property that will be true before and after running the loop.

A loop invariant is

before running loop i , $sum = \sum_{m=1}^{i-1} A[m]$

Example loop invariants with summation

Let us check with some sample input

Input = [9,5,7,4,2]

Initialization:
At i = 1 , m=1-1 = 0
hence, sum = 0
holds True!!!

```
sum = 0
for  i=1 to length[A]
        sum = sum + A[i]
```

Example loop invariants with summation

Maintenance:
If sum (before) = sum from 1 to i-1 then
sum(before next iter) = sum from 1 to i-1 +1

Input = [9,5,7,4,2]

```
sum = 0
for  i=1 to length[A]
        sum = sum + A[i]
```

i	Sum(before) sum to i - 1	Sum(after)
1	0	0 + 9
2	9	0+9+5
3	14	0+9+5+7
4	21	0+9+5+7+4
5	25	0+9+5+7+4+2
6	27	stop

Termination: sum from 1 to n
sum = 0+9+5+7+4+2
Holds True!!

Example loop invariants with summation

Let us check theoretically

```
sum = 0
for  i=1 to length[A]
        sum = sum + A[i]
```

A loop invariant is

before running at loop i , $sum = \sum_{m=1}^{i-1} A[m]$

Initialization: at loop 1, sum = 0 (True!!)

Maintenance:

If at before running loop i , sum = A[1]+A[2]+…+A[i-1]

then after running loop i , sum = A[1]+A[2]+…+A[i-1]+A[i]

Hence, before running loop i+1 , sum = A[1]+A[2]+…+A[i-1]+A[i] (True!!)

Termination:

Goal(output of program) => $sum = \sum_{i=1}^{n} A[i]$

At start of running at loop n+1, sum = A[1]+A[2]+…+A[n-1]+A[n] (True!!)

Exercise: Loop variant with Max Array

- Write a pseudo code of an algorithm for finding a maximal number in an array of size n.
- Write a proof of the correctness of the algorithm using loop invariants.

```
max = A[1]
for  i=2 to length[A]
        if max < A[i]
                max = A[i]
```

Solution: Loop variant with Max Array

```
max = A[1]
for  i=2 to length[A]
        if max < A[i]
                max = A[i]
```

Loop Invariant = Before running loop i , max is the largest number from A[1] to A[i-1]

Initialization:
Before running first loop where i=2 , max = A[1] which is the maximum number of A[2-1] (True!!)

Maintenance:
If before running loop i , max is the largest number among A[1] to A[i-1]
then   after running loop i, if max < A[i] then max = A[i] which is the largest of A[1… i]
if max > A[i] then max does not change and it is the largest of A[1…i].
Hence before running loop i+1, max is is the largest number among A[1] to A[i] (True!!)

Termination: at starting of loop n+1, max is the largest number among A[1] to A[n] (True!!)

Exercise: Insertion-Sort

A loop invariant =
all elements in A[1 … j – 1] are in sorted order.

initialization

```
for j=2 to length[A]
    do key = A[ j ]

    i = j - 1
    while i > 0 and A[ i ] > key
        do A[i+1] = A[ i ]
        i = i - 1
    A[i+1]=key
```

mainte-
nance

termination

Exercise loop invariants with insertion-sort

```
for j=2 to length[A]
        do key = A[ j ]

        i = j - 1
        while i > 0 and A[ i ] > key
                do A[i+1] = A[ i ]
                i = i - 1
        A[i+1]=key
```

Exercise loop invariants with insertion-sort

A loop invariant =
all elements in A[1 … j – 1] are in sorted order.

Input = [9,5,7,4,2]

j	key	A[1 to j-1] (before)	i	A[i] > key	A[1 to j-1] (after)	A[1… n]
2						
3						
4						
5						